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Research Article

On the Asymptoticity Aspect of Hyers-Ulam Stability of Quadratic Mappings

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We investigate the Hyers-Ulam stability of the quadratic functional equation on restricted domains. Applying these results, we study of an asymptotic behavior of these quadratic mappings.

1. Introduction

The question concerning the stability of group homomorphisms was posed by Ulam [1]. Hyers [2] solved the case of approximately additive mappings on Banach spaces. Aoki [3] provided a generalization of the Hyers' theorem for additive mappings. In [4], Rassias generalized the result of Hyers for linear mappings by allowing the Cauchy difference to be unbounded (see also [5]). The result of Rassias has been generalized by Găvruţa [6] who permitted the norm of the Cauchy difference $f(x+y) - f(x) - f(y)$ to be bounded by a general control function under some conditions. This stability concept is also applied to the case of various functional equations by a number of authors. For more results on the stability of functional equations, see [7–32]. We also refer the readers to the books [33–37].

It is easy to see that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = cx^2$ with c an arbitrary constant is a solution of the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

So, it is natural that each equation is called a *quadratic functional* equation. In particular, every solution of the quadratic equation (1.1) is said to be a *quadratic function*. It is well known that a function $f : X \to Y$ between real vector spaces X and Y is quadratic if and only if there exists a unique symmetric biadditive function $B : X \times X \to Y$ such that $f(x) = B(x,x)$ for all $x \in X$ (see [21, 33, 35]).

A stability theorem for the quadratic functional equation (1.1) was proved by Skof [38] for functions $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [11] noticed that the result of Skof holds (with the same proof) if X is replaced by an abelian group G. In [12], Czerwik generalized the result of Skof by allowing growth of the form $\varepsilon \cdot (\|x\|^p + \|y\|^p)$ for the norm of $f(x+y) - f(x-y) - 2f(x) - 2f(y)$, where $\varepsilon > 0$ and $p \neq 2$. In 1998, Jung [39] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains (see also [40–42]). Rassias [43] investigated the Hyers-Ulam stability of mixed type mappings on restricted domains. In [44], the authors considered the asymptoticity of Hyers-Ulam stability close to the asymptotic derivability.

2. Stability of (1.1) on Restricted Domains

In this section, we investigate the Hyers-Ulam stability of the functional equation (1.1) on a restricted domain. As an application, we use the result to the study of an asymptotic behavior of that equation.

Theorem 2.1. *Given a real normed vector space X and a real Banach space Y, let $\varepsilon, \delta, \theta \geq 0$ and $M, p > 0$ with $0 < p < 1$ be fixed. If a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varphi(x,y), \tag{2.1}$$

for all $x, y \in X$ such that $\|x\|^p + \|y\|^p \geq M^p$, where $\varphi(x,y) = \delta + \varepsilon(\|x\|^{2p} + \|y\|^{2p}) + \theta\|x\|^p\|y\|^p$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|Q(x) - f(x)\| \leq \frac{3\delta + M^{2p} \cdot \varepsilon}{6} + \frac{2\varepsilon + \theta}{4 - 4^p}\|x\|^{2p}, \tag{2.2}$$

for all $x \in X$ with $\|x\| \geq M/2^{1/p}$ and $Q(x) = \lim_{n\to\infty}(f(2^n x)/4^n)$. Moreover, if f is measurable or if $f(tx)$ is continuous in t for each fixed $x \in X$, then $Q(tx) = t^2 Q(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. Letting $y = x$ in (2.1), we get

$$\|f(2x) - 4f(x) + f(0)\| \leq \delta + (2\varepsilon + \theta)\|x\|^{2p}, \tag{2.3}$$

for all $x \in X$ with $\|x\| \geq M/2^{1/p}$. If we put $x \in X$ with $\|x\| = M$ and $y = 0$ in (2.1), we obtain

$$\|f(0)\| \leq \frac{\delta + M^{2p} \cdot \varepsilon}{2}. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\|f(2x) - 4f(x)\| \leq \frac{3\delta + M^{2p} \cdot \varepsilon}{2} + (2\varepsilon + \theta)\|x\|^{2p}, \tag{2.5}$$

for all $x \in X$ with $\|x\| \geq M/2^{1/p}$. Replacing x by $2^n x$ in (2.5), we infer the inequality

$$\left\| \frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n} \right\| \leq \frac{3\delta + M^{2p} \cdot \varepsilon}{8 \times 4^n} + \frac{2\varepsilon + \theta}{4} \left(\frac{4^p}{4} \right)^n \|x\|^{2p}, \tag{2.6}$$

for all $x \in X$ with $\|x\| \geq M/2^{1/p}$ and all integers $n \geq 0$. Therefore,

$$\begin{aligned} \left\| \frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^m x)}{4^m} \right\| &\leq \sum_{k=m}^{n} \left\| \frac{f(2^{k+1}x)}{4^{k+1}} - \frac{f(2^k x)}{4^k} \right\| \\ &\leq \frac{3\delta + M^{2p} \cdot \varepsilon}{8} \sum_{k=m}^{n} \frac{1}{4^k} + \frac{2\varepsilon + \theta}{4} \sum_{k=m}^{n} \left(\frac{4^p}{4} \right)^k \|x\|^{2p}, \end{aligned} \tag{2.7}$$

for all $x \in X$ with $\|x\| \geq M/2^{1/p}$ and all integers $n \geq m \geq 0$. It follows from (2.7) that the sequence $\{4^{-n} f(2^n x)\}$ converges for all $x \in X$ with $\|x\| \geq M/2^{1/p}$. Let us denote $\varphi(x) = \lim_{n\to\infty}(f(2^n x)/4^n)$ for all $x \in X$ with $\|x\| \geq M/2^{1/p}$. It is clear that

$$\varphi(2x) = 4\varphi(x), \tag{2.8}$$

for all $x \in X$ with $\|x\| \geq M/2^{1/p}$. Letting $m = 0$ and $n \to \infty$ in (2.7), we get

$$\|\varphi(x) - f(x)\| \leq \frac{3\delta + M^{2p} \cdot \varepsilon}{6} + \frac{2\varepsilon + \theta}{4 - 4^p} \|x\|^{2p}, \tag{2.9}$$

for all $x \in X$ with $\|x\| \geq M/2^{1/p}$.

Now, suppose that $x, y \in X$ such that $\|x\|, \|y\|, \|x \pm y\| \geq M/2^{1/p}$, then by (2.1) and the definition of φ, we obtain

$$\varphi(x+y) + \varphi(x-y) = 2\varphi(x) + 2\varphi(y). \tag{2.10}$$

We have to extend the mapping φ to the whole space X. Given any $x \in X$ with $0 < \|x\| < M/2^{1/p}$, let $k = k(x)$ denote the largest integer such that $M/2^{1/p} \leq 2^k\|x\| < M$. Consider the mapping $Q : X \to Y$ defined by $Q(0) = 0$ and

$$Q(x) = \begin{cases} \dfrac{\varphi(2^k x)}{4^k} & \text{for } 0 < \|x\| < \dfrac{M}{2^{1/p}}, \text{ where } \ k = k(x), \\ \varphi(x) & \text{for } \|x\| \geq \dfrac{M}{2^{1/p}}. \end{cases} \tag{2.11}$$

Let $x \in X$ with $0 < \|x\| < M/2^{1/p}$ and let $k = k(x)$. We have two cases.

Case 1. If $2\|x\| \geq M/2^{1/p}$, we have from (2.8) that

$$Q(2x) = \varphi(2x) = \frac{\varphi(4x)}{4} = \cdots = \frac{\varphi(2^k x)}{4^{k-1}} = 4Q(x). \tag{2.12}$$

Case 2. If $0 < 2\|x\| < M/2^{1/p}$, then $k-1$ is the largest integer satisfying $M/2^{1/p} \le 2^{k-1}\|2x\| < M$, and we have

$$Q(2x) = \frac{\varphi(2^k x)}{4^{k-1}} = 4\frac{\varphi(2^k x)}{4^k} = 4Q(x). \tag{2.13}$$

Therefore, $Q(2x) = 4Q(x)$ for all $x \in X$ with $0 < \|x\| < M/2^{1/p}$. From the definition of Q and (2.8), it follows that $Q(2x) = 4Q(x)$ for all $x \in X$. Now, suppose that $x \in X$ with $x \neq 0$ and choose a positive integer m such that $\|2^m x\| \ge M/2^{1/p}$. By the definition of Q and its property, we have

$$Q(x) = \frac{Q(2^m x)}{4^m} = \frac{\varphi(2^m x)}{4^m}. \tag{2.14}$$

So by the definition of φ, we have

$$Q(x) = \lim_{n\to\infty} \frac{f(2^{m+n}x)}{4^{m+n}} = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}, \tag{2.15}$$

for all $x \in X$ with $x \neq 0$. Since $Q(0) = 0$, (2.15) holds true for $x = 0$. Let $x, y \in X$ with $x, y \neq 0$. It follows from (2.1) and (2.15) that

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y). \tag{2.16}$$

Letting $y = -x$ in (2.16), we get $Q(-x) = Q(x)$ for all $x \in X$ with $x \neq 0$. Since $Q(0) = 0$, the same is true for $x = 0$. So, Q is even and this implies that (2.16) is true for all $x, y \in X$. Therefore, Q is quadratic. By the definition $Q(x) = \varphi(x)$ when $\|x\| \ge M/2^{1/p}$, thus (2.2) follows from (2.9). To prove the uniqueness of Q, let $T : X \to Y$ be another quadratic mapping satisfying (2.2) for all $\|x\| \ge M/2^{1/p}$. Let $x \in X$ with $x \neq 0$ and choose a positive integer m such that $\|2^m x\| \ge M/2^{1/p}$, then

$$\begin{aligned} \|Q(2^n x) - T(2^n x)\| &\le \|Q(2^n x) - f(2^n x)\| + \|f(2^n x) - T(2^n x)\| \\ &\le \frac{M^{2p}\cdot\varepsilon + 12\delta}{12} + \frac{2(2\varepsilon+\theta)4^{np}}{4-4^p}\|x\|^{2p}, \end{aligned} \tag{2.17}$$

for all $n \ge m$. Since Q and T are quadratic, we get

$$\|Q(x) - T(x)\| \le \frac{M^{2p}\cdot\varepsilon + 12\delta}{12\times 4^n} + \frac{2(2\varepsilon+\theta)}{4-4^p}\left(\frac{4^p}{4}\right)^n \|x\|^{2p}, \tag{2.18}$$

for all $n \ge m$. Therefore, $Q(x) = T(x)$. Since $Q(0) = T(0) = 0$, we have $Q(x) = T(x)$ for all $x \in X$. The proof of our last assertion follows from the proof of Theorem 1 in [12]. □

We now introduce one of the fundamental results of fixed point theory by Margolis and Diaz.

Theorem 2.2 (see [22]). *Let (E, d) be a complete generalized metric space and let $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. If there exists a nonnegative integer k such that $d(J^k x, J^{k+1} x) < \infty$ for some $x \in X$, then the following are true:*

(1) *the sequence $\{J^n x\}$ converges to a fixed point x^* of J,*

(2) *x^* is the unique fixed point of J in*

$$Y = \left\{ y \in E : d\left(J^k x, y\right) < \infty \right\}, \tag{2.19}$$

(3) *$d(y, x^*) \le (1/(1-L)) d(y, Jy)$ for all $y \in Y$.*

By using the idea of Cădariu and Radu [45], we applied a fixed point method to the investigation of the generalized Hyers-Ulam stability of the functional equation (1.1) on a restricted domain.

Theorem 2.3. *Given a real normed vector space X and a real Banach space Y, let $M > 0$ be fixed and let $f : X \to Y$ be a mapping which satisfies the inequality (2.1) for all $x, y \in S := \{(x, y) \in X \times X : \|x\|, \|y\|, \|x \pm y\| \ge M\}$, where $\varphi(x, y) : X \times X \to Y$ is a function such that*

$$\varphi(2x, 2y) \le 4L\varphi(x, y), \tag{2.20}$$

for all $x, y \in X$, where $0 < L < 1$ is a constant number, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|Q(x) - f(x)\| \le \frac{1}{1-L}\sigma(x), \tag{2.21}$$

for all $x \in X$ with $\|x\| \ge M$, where

$$\sigma(x) := \frac{1}{8}\left[\varphi(5x, x) + \varphi(4x, 2x) + 2\varphi(4x, x) + 5\varphi(3x, x) + 8\varphi(2x, x)\right] \tag{2.22}$$

and $Q(x) = \lim_{n\to\infty}(f(2^n x)/4^n)$ for all $x \in X$. Moreover, if f is measurable or if $f(tx)$ is continuous in t for each fixed $x \in X$, then $Q(tx) = t^2 Q(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. It follows from (2.20) that

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{4^n} = 0, \tag{2.23}$$

for all $x, y \in X$. Let $y \in X_M := \{x \in X : \|x\| \geq M\}$. Letting $x = ky$ for $k = 2, 3, 4, 5$ in (2.1), we get the following inequalities:

$$\|f(3y) - 2f(2y) - f(y)\| \leq \varphi(2y, y), \tag{2.24}$$

$$\|f(4y) - 2f(3y) + f(2y) - 2f(y)\| \leq \varphi(3y, y), \tag{2.25}$$

$$\|f(5y) - 2f(4y) + f(3y) - 2f(y)\| \leq \varphi(4y, y), \tag{2.26}$$

$$\|f(6y) - 2f(5y) + f(4y) - 2f(y)\| \leq \varphi(5y, y). \tag{2.27}$$

It follows from (2.24) and (2.25) that

$$\|f(4y) - 3f(2y) - 4f(y)\| \leq 2\varphi(2y, y) + \varphi(3y, y). \tag{2.28}$$

By (2.26) and (2.27), we have

$$\|f(6y) - 3f(4y) + 2f(3y) - 6f(y)\| \leq 2\varphi(4y, y) + \varphi(5y, y). \tag{2.29}$$

It follows from (2.25) and (2.29) that

$$\|f(6y) - 2f(4y) + f(2y) - 8f(y)\| \leq \varphi(5y, y) + 2\varphi(4y, y) + \varphi(3y, y). \tag{2.30}$$

Using (2.28) and (2.30), we have

$$\|f(6y) - 5f(2y) - 16f(y)\| \leq \varphi(5y, y) + 2\varphi(4y, y) + 3\varphi(3y, y) + 4\varphi(2y, y). \tag{2.31}$$

By (2.24), we get

$$\|f(6y) - 2f(4y) - f(2y)\| \leq \varphi(4y, 2y). \tag{2.32}$$

Hence, we obtain from (2.31) and (2.32) that

$$\|2f(4y) - 4f(2y) - 16f(y)\| \leq \varphi(5y, y) + \varphi(4y, 2y) + 2\varphi(4y, y) + 3\varphi(3y, y) + 4\varphi(2y, y). \tag{2.33}$$

So, it follows from (2.28) and (2.33) that

$$\left\| \frac{f(2y)}{4} - f(y) \right\| \leq \sigma(y), \tag{2.34}$$

for all $y \in X_M$. Let $E := \{h : X_M \to Y\}$. We introduce a generalized metric on E as follows:

$$d(h, k) := \inf\{C \in [0, \infty] : \|h(x) - k(x)\| \leq C\sigma(x)\ \forall x \in X_M\ \}. \tag{2.35}$$

We assert that (E, d) is a generalized complete metric space. Let $\{h_n\}$ be a Cauchy sequence in (E, d) and $\varepsilon > 0$ be given, then there exists an integer N such that $d(h_m, h_n) \leq \varepsilon$ for all $m, n \geq N$. This implies that $\|h_m(x) - h_n(x)\| \leq \varepsilon\sigma(x)$ for all $x \in X_M$ and all $m, n \geq N$. Therefore, $\{h_n(x)\}$ is a Cauchy sequence in Y for all $x \in X_M$. Since Y is a Banach space, $\{h_n(x)\}$ converges for all $x \in X_M$. Thus, we can define a function $h : X_M \to Y$ by

$$h(x) := \lim_{n\to\infty} h_n(x). \tag{2.36}$$

Since

$$\|h_m(x) - h(x)\| = \lim_{n\to\infty} \|h_m(x) - h_n(x)\| \leq \varepsilon\sigma(x), \tag{2.37}$$

for all $x \in X_M$ and all $m \geq N$, we get $d(h_m, h) \leq \varepsilon$ for all $m \geq N$. That is, the Cauchy sequence $\{h_n\}$ converges to h in (E, d). Hence, (E, d) is complete. We now consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda h)(x) = \frac{1}{4}h(2x), \quad \forall h \in E,\ x \in X_M. \tag{2.38}$$

Let $h, k \in E$ and let $C \in [0, \infty]$ be an arbitrary constant with $d(h, k) \leq C$. From the definition of d, we have

$$\|h(x) - k(x)\| \leq C\sigma(x), \tag{2.39}$$

for all $x \in X_M$. By the assumption (2.20) and the last inequality, we have

$$\|(\Lambda h)(x) - (\Lambda k)(x)\| = \frac{1}{4}\|h(2x) - k(2x)\| \leq \frac{C}{4}\sigma(2x) \leq CL\sigma(x), \tag{2.40}$$

for all $x \in X_M$. So $d(\Lambda h, \Lambda k) \leq Ld(h, k)$. That is, Λ is a strictly contractive on E. It follows from (2.34) that $d(\Lambda f, f) \leq 1$. Therefore, according to Theorem 2.2, there exists a function $\varphi \in E$ such that the sequence $\{\Lambda^n f\}$ converges to φ and $\Lambda\varphi = \varphi$. Indeed,

$$\varphi : X_M \to Y, \quad \varphi(x) = \lim_{n\to\infty} (\Lambda^n f)(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \tag{2.41}$$

and $\varphi(2x) = 4\varphi(x)$, for all $x \in X_M$. Also, φ is the unique fixed point of Λ in the set $E^* = \{h \in E : d(f, h) < \infty\}$ and

$$d(\varphi, f) \leq \frac{1}{1-L}d(\Lambda f, f) \leq \frac{1}{1-L}. \tag{2.42}$$

By (2.1), (2.23) and using the definition of φ, we get

$$\varphi(x + y) + \varphi(x - y) = 2\varphi(x) + 2\varphi(y), \tag{2.43}$$

for all $(x, y) \in S$. We will define a mapping $Q : X \to Y$ such that $Q|_{X_M} = \varphi$. Similar to the proof of Theorem 2.1 for a given $x \in X$ with $0 < \|x\| < M$, let $k = k(x)$ denote the largest integer such that $M/2 \le 2^k\|x\| < M$. Consider the mapping $Q : X \to Y$ defined by $Q(0) = 0$ and

$$Q(x) = \begin{cases} \dfrac{\varphi(2^k x)}{4^k} & \text{for } 0 < \|x\| < M, \text{ where } k = k(x), \\ \varphi(x) & \text{for } \|x\| \ge M. \end{cases} \tag{2.44}$$

Let $x \in X$ with $0 < \|x\| < M$ and let $k = k(x)$. We have two cases.

Case 1. $2\|x\| \ge M$. Since $\varphi(2x) = 4\varphi(x)$ for all $x \in X_M$, we have

$$Q(2x) = \varphi(2x) = \frac{\varphi(4x)}{4} = \cdots = \frac{\varphi(2^k x)}{4^{k-1}} = 4Q(x). \tag{2.45}$$

Case 2. If $0 < 2\|x\| < M$, then $k-1$ is the largest integer satisfying $M/2 \le 2^{k-1}\|2x\| < M$, and we have

$$Q(2x) = \frac{\varphi(2^k x)}{4^{k-1}} = 4\frac{\varphi(2^k x)}{4^k} = 4Q(x). \tag{2.46}$$

Therefore, $Q(2x) = 4Q(x)$ for all $x \in X$ with $0 < \|x\| < M$. Using $\varphi(2x) = 4\varphi(x)$ for all $x \in X_M$ and the definition of Q, we get that $Q(2x) = 4Q(x)$ for all $x \in X$. Now, suppose that $x \in X$ with $x \ne 0$ and choose a positive integer m such that $\|2^m x\| \ge M$. By the definition of Q and its property, we have

$$Q(x) = \frac{Q(2^m x)}{4^m} = \frac{\varphi(2^m x)}{4^m}. \tag{2.47}$$

So by the definition of φ, we have

$$Q(x) = \lim_{n\to\infty} \frac{f(2^{m+n}x)}{4^{m+n}} = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}, \tag{2.48}$$

for all $x \in X$ with $x \ne 0$. Since $Q(0) = 0$, (2.48) holds true for $x = 0$. Let $x, y \in X$ with x, y, $x \pm y \ne 0$. It follows from (2.1), (2.23), and (2.48) that

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y). \tag{2.49}$$

Since $Q(0) = 0$ and $Q(2x) = 4Q(x)$ for all $x \in X$, we conclude that (2.49) is true for all $y \in \{0, x\}$. Let $y \in X$ with $y \ne 0$. Putting $x = 2y$ in (2.49), we get $Q(3y) = 9Q(y)$. Therefore, by letting $y = 2x$ in (2.49), we get $Q(-x) = Q(x)$ for all $x \in X$ with $x \ne 0$. Since $Q(0) = 0$, the same is true for $x = 0$. So, Q is even and this implies that (2.49) is true for all $x, y \in X$. Therefore, Q is quadratic. To prove the uniqueness of Q, let $T : X \to Y$ be another quadratic mapping satisfying (2.21), for all $\|x\| \ge M$. Let $x \in X$ with $x \neq 0$ and choose a positive integer m such that $\|2^m x\| \ge M$, then

$$\begin{aligned} \|Q(2^n x) - T(2^n x)\| &\le \|Q(2^n x) - f(2^n x)\| + \|f(2^n x) - T(2^n x)\| \\ &\le \frac{2}{1-L}\sigma(2^n x), \end{aligned} \tag{2.50}$$

for all $n \ge m$. Since Q and T are quadratic, we get

$$\|Q(x) - T(x)\| \le \frac{2}{1-L} \times \frac{\sigma(2^n x)}{4^n}, \tag{2.51}$$

for all $n \ge m$. Therefore, (2.23) implies that $Q(x) = T(x)$. Since $Q(0) = T(0) = 0$, we have $Q(x) = T(x)$ for all $x \in X$. Our last assertion is trivial in view of Theorem 2.1. □

Corollary 2.4. *Given a real normed vector space X and a real Banach space Y, let $\varepsilon, \delta, \theta \ge 0$ and $M, p > 0$ with $0 < p < 1$ be fixed. Suppose that a mapping $f : X \to Y$ satisfies the inequality* (2.1) *for all $(x, y) \in S$, then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\begin{aligned} \|Q(x) - f(x)\| \le \frac{1}{2(4-4^p)} &[17\delta + (25^p + 3 \times 16^p + 5 \times 9^p + 9 \times 4^p + 16)\varepsilon \\ &+(8^p + 5^p + 2 \times 4^p + 5 \times 3^p + 8 \times 2^p)\theta] \|x\|^{2p}, \end{aligned} \tag{2.52}$$

for all $x \in X$ with $\|x\| \ge M$ and $Q(x) = \lim_{n\to\infty}(f(2^n x)/4^n)$. Moreover, if f is measurable or if $f(tx)$ is continuous in t for each fixed $x \in X$, then $Q(tx) = t^2 Q(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Remark 2.5. We may replace the condition (2.20) by

$$\begin{aligned} &\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{4^n} = 0 \quad (x, y) \in S, \\ &\widetilde{\varphi}(x, y) := \sum_{n=1}^{\infty} \frac{\varphi(2^n x, 2^n y)}{4^n} < \infty, \end{aligned} \tag{2.53}$$

for all $y \in X$ and $x \in \{2y, 3y, 4y, 5y\}$. Using the direct method, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|Q(x) - f(x)\| \le \frac{1}{8}\left[\widetilde{\varphi}(5x, x) + \widetilde{\varphi}(4x, 2x) + 2\widetilde{\varphi}(4x, x) + 5\widetilde{\varphi}(3x, x) + 8\widetilde{\varphi}(2x, x)\right], \tag{2.54}$$

for all $x \in X$ with $\|x\| \geq M$. For the case $\psi(x,y) = \delta + \varepsilon(\|x\|^{2p} + \|y\|^{2p}) + \theta\|x\|^p\|y\|^p$, where $\delta, \varepsilon, \theta \geq 0$ and $0 < p < 1$, we have

$$\begin{aligned}\|Q(x) - f(x)\| \leq \frac{17}{6}\delta + \frac{1}{2(4-4^p)}\big[&(25^p + 3\times 16^p + 5\times 9^p + 9\times 4^p + 16)\varepsilon \\ &+(8^p + 5^p + 2\times 4^p + 5\times 3^p + 8\times 2^p)\theta\big]\|x\|^{2p}.\end{aligned} \tag{2.55}$$

Using ideas from the papers [39, 43], we prove the generalized Hyers-Ulam stability of (1.1) on restricted domains. We first prove the following lemma.

Lemma 2.6. *Given a real normed vector space X and a real Banach space Y, let $M, p > 0$ and $\delta, \varepsilon \geq 0$ be fixed. If a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta + \varepsilon(\|x\|^p + \|y\|^p), \tag{2.56}$$

for all $x, y \in X$ with $\|x\|^p + \|y\|^p \geq M^p$, then

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y) - f(0)\| \leq \phi(x,y), \tag{2.57}$$

for all $x, y \in X$, where

$$\phi(x,y) := \frac{1}{2}\Big[9\delta + (16^p + 4\times 9^p + 8\times 4^p)M^{2p}\varepsilon + \varepsilon(\|x-y\|^p + 2\|x\|^p + 2\|y\|^p)\Big]. \tag{2.58}$$

Proof. Assume that $\|x\|^p + \|y\|^p < M^p$. If $x = y = 0$, then we choose a $t \in X$ with $\|t\| = M$. Otherwise, let

$$t = \begin{cases} (\|x\| + M)\dfrac{x}{\|x\|} & \text{if } \|x\| \geq \|y\|, \\ (\|y\| + M)\dfrac{y}{\|y\|} & \text{if } \|y\| \geq \|x\|. \end{cases} \tag{2.59}$$

It is clear that $\|t\| \geq M$ and

$$\begin{gathered}\|x-t\|^p + \|y+t\|^p \geq \max\{\|x-t\|^p, \|y+t\|^p\} \geq M^p, \\ \|x-y\|^p + \|2t\|^p \geq \|t\|^p \geq M^p, \\ \|x+t\|^p + \|t-y\|^p \geq \max\{\|x+t\|^p, \|t-y\|^p\} \geq M^p, \\ \min\{\|x\|^p + \|t\|^p, \|y\|^p + \|t\|^p, \|t\|^p + \|t\|^p\} \geq \|t\|^p \geq M^p.\end{gathered} \tag{2.60}$$

Also

$$\max\{\|x-t\|, \|x+t\|, \|y-t\|, \|y+t\|\} < 3M, \quad \|t\| < 2M. \tag{2.61}$$

Therefore,

$$\begin{aligned}
&2\left[f(x+y)+f(x-y)-2f(x)-2f(y)-f(0)\right] \\
&\quad=\left[f(x+y)+f(x-y-2t)-2f(x-t)-2f(y+t)\right] \\
&\qquad-\left[f(x-y-2t)+f(x-y+2t)-2f(x-y)-2f(2t)\right] \\
&\qquad+\left[f(x-y+2t)+f(x+y)-2f(x+t)-2f(t-y)\right] \\
&\qquad+2\left[f(x+t)+f(x-t)-2f(x)-2f(t)\right] \\
&\qquad+2\left[f(t+y)+f(t-y)-2f(t)-2f(y)\right] \\
&\qquad-2\left[f(2t)+f(0)-2f(t)-2f(t)\right].
\end{aligned} \tag{2.62}$$

So, we get

$$\begin{aligned}
&2\|f(x+y)+f(x-y)-2f(x)-2f(y)-f(0)\| \\
&\quad\le 9\delta+(16^p+4\times 9^p+8\times 4^p)M^{2p}\varepsilon+\varepsilon\left(\|x-y\|^p+2\|x\|^p+2\|y\|^p\right).
\end{aligned} \tag{2.63}$$

So, f satisfies (2.57) for all $x,y\in X$. □

Theorem 2.7. *Given a real normed vector space X and a real Banach space Y, let $\delta,\varepsilon\ge 0$ and $M,p>0$ with $0<p<2$ be given. Assume that a mapping $f:X\to Y$ satisfies the inequality* (2.56) *for all $x,y\in X$ with $\|x\|^p+\|y\|^p\ge M^p$, then there exists a unique quadratic mapping $Q:X\to Y$ such that $Q(x)=\lim_{n\to\infty}4^{-n}f(2^nx)$ and*

$$\|f(x)-Q(x)\|\le\frac{1}{6}\left[9\delta+(16^p+4\times 9^p+8\times 4^p)M^{2p}\varepsilon\right]+\frac{2\varepsilon}{4-2^p}\|x\|^p, \tag{2.64}$$

for all $x\in X$.

Proof. By Lemma 2.6, f satisfies (2.57) for all $x,y\in X$. Letting $y=x$ in (2.57), we get

$$\left\|\frac{f(2x)}{4}-f(x)\right\|\le K+\frac{\varepsilon}{2}\|x\|^p, \tag{2.65}$$

for all $x\in X$, where

$$K:=\frac{1}{8}\left[9\delta+(16^p+4\times 9^p+8\times 4^p)M^{2p}\varepsilon\right]. \tag{2.66}$$

We can use the argument given in the proof of Theorem 2.1 to arrive the inequality

$$\left\|\frac{f(2^{n+1}x)}{4^{n+1}}-\frac{f(2^mx)}{4^m}\right\|\le K\sum_{k=m}^{n}\frac{1}{4^k}+\frac{\varepsilon}{2}\sum_{k=m}^{n}\left(\frac{2^p}{4}\right)^k\|x\|^p, \tag{2.67}$$

for all $x \in X$ and all integers $n \geq m \geq 0$. It follows from (2.67) that the sequence $\{4^{-n} f(2^n x)\}$ converges for all $x \in X$. So, we can define the mapping $Q : X \to Y$ by $Q(x) = \lim_{n\to\infty}(f(2^n x)/4^n)$ for all $x \in X$. Letting $m = 0$ and $n \to \infty$ in (2.67), we get (2.64). □

For the case $\varepsilon = 0$ and $p = 1$ in Theorem 2.7, it is obvious that our inequality (2.64) is sharper than the corresponding inequalities of Jung [39] and Rassias [43].

Skof [38] has proved an asymptotic property of the additive mappings, and Jung [39] has proved an asymptotic property of the quadratic mappings (see also [41]). Using the method in [39], the proof of the following corollary follows from Theorem 2.7 by letting $\varepsilon = 0$ and $p = 1$.

Corollary 2.8 (see [39]). *Given a real normed vector space X and a real Banach space Y, a mapping $f : X \to Y$ satisfies* (1.1) *if and only if the asymptotic condition*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \longrightarrow 0 \quad \text{as } \|x\| + \|y\| \longrightarrow \infty \tag{2.68}$$

holds true.

3. p-Asymptotically Quadratic Mappings

We apply our results to the study of p-asymptotical derivatives. Let X be a real normed vector space and let Y be a real Banach space Y. Let $0 < p < 2$ be arbitrary.

Definition 3.1. A mapping $f : X \to Y$ is called *p-asymptotically close* to a mapping $T : X \to Y$ if and only if $\lim_{\|x\|\to\infty}(\|f(x) - T(x)\|/\|x\|^p) = 0$.

Definition 3.2. A mapping $f : X \to Y$ is called *p-asymptotically derivable* if the mapping f is p-asymptotically close to a quadratic mapping $Q : X \to Y$. In this case, we say that Q is a p-asymptotical derivative of f.

Definition 3.3. A mapping $f : X \to Y$ is called *p-asymptotically quadratic* if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \tag{3.1}$$

for all $x, y \in X$ with $\|x\|, \|y\|, \|x \pm y\| \geq \delta$.

Definition 3.4. A mapping $T : X \to Y$ is called *quadratic outside a ball* if there exists $\delta > 0$ such that $T(x+y) + T(x-y) = 2T(x) + 2T(y)$ for all $x, y \in X$ with $\|x\|, \|y\|, \|x \pm y\| \geq \delta$.

We have the following result.

Theorem 3.5. *If $T : X \to Y$ is quadratic outside a ball and $f : X \to Y$ is p-asymptotically close to T, then f is p-asymptotically quadratic.*

The following result follows from Corollary 2.4.

Corollary 3.6. *If $T : X \to Y$ is quadratic outside a ball and $f : X \to Y$ is p-asymptotically close to T, then f has a p-asymptotical derivative.*

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